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LARGE MIXING ANGLES IN A $SU(2)_L$ GAUGE THEORY OF WEAK INTERACTIONS AS A RESONANT EFFECT OF 1-LOOP TRANSITIONS BETWEEN QUASI-DEGENERATE FERMIONS

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Abstract: We show that 1-loop transitions between two quasi-degenerate fermions can induce a potentially large renormalization of their mixing angle, and a large renormalized Cabibbo (or PMNS) angle when the second fermion pair in the same two generations is far from degeneracy. At the resonance, the "Cabibbo angle" gets maximal and simply connected to masses without invoking any new physics beyond the standard model. This solution appears as the only one "perturbatively stable" (mixing angles are then renormalized with respect to their classical values by small amounts).

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1 Introduction

The origin of large mixing angles observed in leptonic charged currents is still unknown [1]. A common idea is that it is linked to a quasi-degeneracy of neutrinos, but this connection was never firmly established. And it cannot be on simple grounds, since homographic transformations on a (mass) matrix, while changing its eigenvalues, do not change its eigenvectors, hence mixing angles; accordingly, infinitely different spectra can be associated to a given mixing angle.

We show below, in the case of binary coupled systems, that large mixing can be associated with quasi-degeneracy. Indeed, small (perturbative) changes of parameters (for examples elements of mass matrix) can then trigger large variations of eigenstates. In the case under scrutiny, 1-loop transitions between two fermions generate perturbative $\mathcal{O}(g^2)$ modifications of their kinetic terms. A (slightly non-unitary) transformation, which differs from a rotation only at $\mathcal{O}(g^2)$, is needed to cast them back into their canonical form $\overline{\Psi}\,\not{p}\,\mathbb{I}\,\Psi$ ($\mathbb{I}$ is the unit matrix). When the two fermions are quasi-degenerate, the induced transformation of their mass matrix is enough to trigger in turn large variations of its eigenvectors, such that its re-diagonalization requires a rotation by a large angle. The latter ultimately becomes the renormalized Cabibbo angle that occurs in charged current.

In the following, we shall work with two generations of fermions, and take the example of two pairs of quarks (d,s) and (u,c). This can be easily translated to the (more appropriate) lepton case, when the two pairs are instead, for example, (ν_e,ν_μ) and (e^-,μ^-). Then, "Cabibbo angle" [2] translates into "first PMNS angle" θ_{12} [3], "quasi-degenerate (d,s) system" into "quasi-degenerate neutrino pair", (u,c) far from degeneracy into $(electron, muon)$ far from degeneracy *etc*. Also, the the sake of simplicity, we shall work in a pure $SU(2)_L$ theory of weak interactions instead of the standard $SU(2)_L\times U(1)$ electroweak model.

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2 1-loop transitions between non-degenerate fermions. Re-diagonalizing the quadratic Lagrangian

2.1 1-loop transitions

Like in the Standard Model of electroweak interactions [4], the diagonalization of the classical mass matrix by a bi-unitary transformation leads to the classical mass eigenstates, for example s_m^0 and d_m^0, with classical masses m_s and m_d. They are orthogonal with respect to the classical Lagrangian (no transition between them occurs at the classical level). However, at 1-loop, gauge interactions induce diagonal and non-diagonal transitions between them. For example, Fig. 1 describes non-diagonal $s_m^0 \rightarrow d_m^0$ transitions [1], mediated by the $W^\pm$ gauge bosons. Diagonal transitions are mediated either by $W_\mu^\pm$ or by the W_μ^3 gauge bosons.

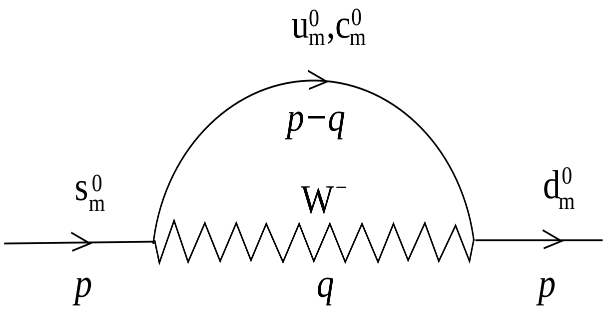

Fig. 1: $s_m^0 \rightarrow d_m^0$ transition at 1-loop

We investigate in this work how the Cabibbo procedure implements in the presence of these transitions [5]. The one depicted in Fig. 1 contributes as a left-handed, kinetic-like, p^2-dependent interaction of the type

$$\sin\theta_c \cos\theta_c \big(h(p^2, m_u, m_W) - h(p^2, m_c, m_W)\big)\, \bar d_m^0 \not p (1-\gamma_5)\, s_m^0, \tag{1}$$

that we abbreviate, with transparent notations, into

$$s_c c_c (h_u - h_c)\, \bar d_m^0 \not p\, (1-\gamma_5)\, s_m^0. \tag{2}$$

It depends on the classical Cabibbo angle $\theta_c = \theta_d - \theta_u$. The function h is dimensionless. It is simple matter to realize that all (diagonal and non-diagonal) 1-loop transitions mediated between s and d mediated by $W^\pm$ gauge bosons transform their kinetic terms into

$$\begin{aligned} & \begin{pmatrix} \bar d_m^0 & \bar s_m^0 \end{pmatrix} \left[\mathbb{I} \not p + \begin{pmatrix} c_c^2 h_u + s_c^2 h_c & s_c c_c (h_u - h_c) \\ s_c c_c (h_u - h_c) & s_c^2 h_u + c_c^2 h_c \end{pmatrix} \not p (1-\gamma_5) \right] \begin{pmatrix} d_m^0 \\ s_m^0 \end{pmatrix} \\ = \quad & \begin{pmatrix} \bar d_m^0 & \bar s_m^0 \end{pmatrix} \left[\mathbb{I} \not p + \left(\frac{h_u + h_c}{2} + (h_u - h_c)\, \mathcal{T}_x(2\theta_c) \right) \not p (1-\gamma_5) \right] \begin{pmatrix} d_m^0 \\ s_m^0 \end{pmatrix}, \end{aligned} \tag{3}$$

where we noted

$$\mathcal{T}_x(\theta) = \frac{1}{2} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{4}$$

To the contributions (3) we must add the diagonal transitions mediated by the W_μ^3 gauge boson. The kinetic terms for left-handed d_m^0 and s_m^0 quarks become (omitting the fermionic fields)

[1] Similar transitions occur between c and u quarks, and between their leptonic equivalent.

$$
\begin{aligned}
K_d &= \mathbb{I} + H_d\ ;\\
H_d &= \frac{h_u+h_c}{2} + (h_u-h_c)\ \mathcal{T}_x(2\theta_c) + \begin{pmatrix} h_d & \\ & h_s \end{pmatrix},
\end{aligned}
\tag{5}
$$

where $h_d = h(p^2, m_d, m_W)$ and $h_s = h(p^2, m_s, m_W)$. Likewise, in the (u,c) sector, one has

$$
\begin{aligned}
K_u &= \mathbb{I} + H_u\ ;\\
H_u &= \frac{h_d+h_s}{2} + (h_d-h_s)\ \mathcal{T}_x(2\theta_c) + \begin{pmatrix} h_u & \\ & h_c \end{pmatrix}.
\end{aligned}
\tag{6}
$$

We shall now diagonalize the quadratic part of the effective 1-loop Lagrangian, which means putting the pure kinetic terms back to the unit matrix and, at the same time, re-diagonalizing the mass matrix. This is accordingly a two-steps procedure.

Note that the kinetic terms of right-handed fermions are not modified, such that we shall only be concerned below with the left-handed ones.

2.2 First step: re-diagonalizing kinetic terms back to the unit matrix

The pure kinetic terms K_d for (d_m^0, s_m^0) written in (5) can be cast back to their canonical form by a p^2-dependent non-unitary transformations $\mathcal{V}_d$ according to

$$
\mathcal{V}_d^\dagger\ K_d\ \mathcal{V}_d = \mathbb{I}. \tag{7}
$$

The procedure to find $\mathcal{V}_d$ is the following. Let $(1+t_+)$ and $(1+t_-)$, $t_+, t_- = \mathcal{O}(g^2)$, be the eigenvalues of the symmetric matrix K_d [2]. It can be diagonalized by a rotation $\mathcal{R}(\omega_d) \equiv \begin{pmatrix} \cos\omega_d & \sin\omega_d \\ -\sin\omega_d & \cos\omega_d \end{pmatrix}$ according to

$$
\mathcal{R}(\omega_d)^\dagger\ K_d\ \mathcal{R}(\omega_d) = \begin{pmatrix} 1+t_+ & \\ & 1+t_- \end{pmatrix}, \tag{9}
$$

with

$$
\tan 2\omega_d = \frac{-(h_u-h_c)\sin 2\theta_c}{(h_u-h_c)\cos 2\theta_c + h_d - h_s} = \frac{-1}{1+\frac{h_d-h_s}{h_u-h_c}}\ \tan 2\theta_c. \tag{10}
$$

(10) defines ω_d in particular as a function of the classical θ_c: $\omega_d = \omega_d(\theta_c, \ldots)$.

The diagonal matrix obtained in (9) is not yet the unit matrix, but one gets to it by a simple renormalization of the columns of $\mathcal{R}(\omega_d)$ respectively by $\frac{1}{\sqrt{1+t_+}}$ and $\frac{1}{\sqrt{1+t_-}}$. The looked for non-unitary matrix $\mathcal{V}_d$ writes finally

$$
\mathcal{V}_d(p^2, \ldots) = \begin{pmatrix} \dfrac{c_{\omega_d}}{\sqrt{1+t_+}} & \dfrac{s_{\omega_d}}{\sqrt{1+t_-}} \\ -\dfrac{s_{\omega_d}}{\sqrt{1+t_+}} & \dfrac{c_{\omega_d}}{\sqrt{1+t_-}} \end{pmatrix}. \tag{11}
$$

---

[2] One has explicitly

$$
t_\pm = \frac{h_u+h_c+h_d+h_s}{2} \pm \frac{1}{2}\sqrt{(h_u-h_c)^2 + (h_d-h_s)^2 + 2\,(h_u-h_c)\,(h_d-h_s)\cos 2\theta_c}. \tag{8}
$$

.

It differs from the rotation $\mathcal{R}(\omega_d)$ only at $\mathcal{O}(g^2)$ and satisfies

$$\mathcal{V}_d\,\mathcal{V}_d^\dagger = \frac{1}{(1+t_+)(1+t_-)}\left(\mathbb{I} + \frac{t_+ + t_-}{2} - (t_+ - t_-)\;\mathcal{T}_x(-2\omega_d)\right), \quad \mathcal{V}_d^\dagger\,\mathcal{V}_d = \begin{pmatrix} \dfrac{1}{1+t_+} & \\ & \dfrac{1}{1+t_-} \end{pmatrix}. \tag{12}$$

For $|h_d - h_s| \ll |h_u - h_c|$, eq. (10) shows that $\omega_d(\theta_c) \approx -\theta_c$. So, when the pair (d,s) is close to degeneracy and (u,c) far from it (see also footnote 6), $\mathcal{V}_d$ becomes close to a rotation $\mathcal{R}(-\theta_c)$. This property plays, as shown in subsection 4.2, an important role in the determination of the renormalized Cabibbo angle.

2.3 Second step: re-diagonalization of the mass matrix

By the flavor transformation $\mathcal{V}_d$ acting on left-handed fermions in the bare mass basis, M_d transforms into $\mathcal{V}_d^\dagger M_d$, which needs to be re-diagonalized. To this purpose, a new bi-unitary transformation is needed. The transformation acting on left-handed fermions is the rotation $\mathcal{R}(\xi_d)$ that diagonalizes the real symmetric matrix [3]

$$\begin{aligned} \mathcal{V}_d^\dagger\,M_d M_d^\dagger\,\mathcal{V}_d &= \mathcal{V}_d^\dagger \begin{pmatrix} m_d^2 & \\ & m_s^2 \end{pmatrix} \mathcal{V}_d \\ &= \begin{pmatrix} \dfrac{m_d^2\,c_{\omega_d}^2 + m_s^2 s_{\omega_d}^2}{1+t_+} & -\dfrac{s_{\omega_d}c_{\omega_d}(m_s^2 - m_d^2)}{\sqrt{(1+t_+)(1+t_-)}} \\ -\dfrac{s_{\omega_d}c_{\omega_d}(m_s^2 - m_d^2)}{\sqrt{(1+t_+)(1+t_-)}} & \dfrac{m_d^2 s_{\omega_d}^2 + m_s^2 c_{\omega_d}^2}{1+t_-} \end{pmatrix}, \end{aligned} \tag{13}$$

according to [4]

$$\mathcal{R}(\xi_d)^\dagger\;\left(\mathcal{V}_d^\dagger\,M_d M_d^\dagger\,\mathcal{V}_d\right)\;\mathcal{R}(\xi_d) = \begin{pmatrix} \mu_d^2 & \\ & \mu_s^2 \end{pmatrix}. \tag{14}$$

It satisfies

$$\tan 2\xi_d = \frac{-(m_d^2 - m_s^2)\sqrt{(1+t_+)(1+t_-)}\,\sin 2\omega_d}{(m_d^2 - m_s^2)\left(1 + \dfrac{t_+ + t_-}{2}\right)\cos 2\omega_d - (m_d^2 + m_s^2)\dfrac{t_+ - t_-}{2}}. \tag{15}$$

(15) defines in particular ξ_d as a function of ω_d, and thus as a function of the classical θ_c: $\xi_d = \xi_d(\theta_c, \ldots)$. It also defines 1-loop mass eigenstates $d_{mL}(p^2)$ and $s_{mL}(p^2)$. Since it is in particular unitary, it preserves the canonical form of the kinetic terms that had been recovered in the first step of the procedure. By construction, at any given p^2, there is no 1-loop transition between $d_{mL}(p^2)$ and $s_{mL}(p^2)$.

The main property of (15) is the presence of a pole. It occurs for

$$2\,\frac{m_d^2 - m_s^2}{m_d^2 + m_s^2}\,\cos 2\omega_d(\theta_c) \approx t_+ - t_- \overset{(8)}{=} \sqrt{(h_u - h_c)^2 + (h_d - h_s)^2 + 2\,(h_u - h_c)(h_d - h_s)\,\cos 2\theta_c}, \tag{16}$$

which is, through (10), a relation between $\theta_c, m_d, m_s, m_u, m_c, m_W$ (and p^2).

We shall see in subsection 4.2 that, for quasi-degenerate (d,s) and largely split (u,c), $\xi_d(\theta_c)$ ultimately becomes the renormalized Cabibbo angle, which is accordingly implicitly expressed by (15) as a function of the masses of fermions and gauge fields, and of p^2.

[3] From now onwards, to lighten the notations, we shall frequently omit the dependence on p^2 and on the masses.

[4] The re-diagonalization of kinetic terms indirectly contributes to a renormalization of the masses: $m_d \to \mu_d, m_s \to \mu_s$.

3 Individual mixing matrix and renormalized mixing angle

3.1 1-loop and classical mass eigenstates are non-unitarily related

The left-handed [5] 1-loop mass eigenstates are related to the bare ones by

$$\begin{pmatrix} d^0_{mL} \\ s^0_{mL} \end{pmatrix} = \mathcal{V}_d\, \mathcal{R}(\xi_d) \begin{pmatrix} d_{mL} \\ s_{mL} \end{pmatrix}. \tag{17}$$

They are thus deduced from the latter by the product of a p^2-dependent non-unitary transformation $\mathcal{V}_d$ and a p^2-dependent unitary one $\mathcal{R}(\xi_d)$. The two basis are accordingly non-unitarily related [6]. In particular, on mass-shell (respectively at $p^2 = m_d^2$ and $p^2 = m_s^2$), one has for the physical mass eigenstates

$$\begin{aligned} d^{phys}_{mL} \equiv d_{mL}(p^2 = m_d^2) &= [\mathcal{V}_d\, \mathcal{R}(\xi_d)]_{11}(p^2 = m_d^2)\; d^0_{mL} + [\mathcal{V}_d\, \mathcal{R}(\xi_d)]_{12}(p^2 = m_d^2)\; s^0_{mL}, \\ s^{phys}_{mL} \equiv s_{mL}(p^2 = m_s^2) &= [\mathcal{V}_d\, \mathcal{R}(\xi_d)]_{21}(p^2 = m_s^2)\; d^0_{mL} + [\mathcal{V}_d\, \mathcal{R}(\xi_d)]_{22}(p^2 = m_s^2)\; s^0_{mL}. \end{aligned} \tag{18}$$

Since bare mass states are unitarily related to bare flavor states, the physical mass eigenstates are also non-unitarily related to bare flavor states.

3.2 Individual mixing matrix and renormalized mixing angle

Classical flavor eigenstates and 1-loop mass eigenstates are related to each other according to

$$\begin{pmatrix} d^0_{fL} \\ s^0_{fL} \end{pmatrix} = \mathcal{C}_{d0} \begin{pmatrix} d^0_{mL} \\ s^0_{mL} \end{pmatrix} \stackrel{(17)}{=} \mathcal{C}_{d0}\, \mathcal{V}_d\, \mathcal{R}(\xi_d) \begin{pmatrix} d_{mL} \\ s_{mL} \end{pmatrix}, \tag{19}$$

where $\mathcal{C}_{d0} \equiv \mathcal{R}(\theta_d)$ is the classical mixing matrix in the (d, s) sector. The individual mixing matrix at 1-loop is thus given by

$$\mathcal{C}_d = \mathcal{C}_{d0}\, \mathcal{V}_d\, \mathcal{R}(\xi_d) = \mathcal{R}(\theta_d)\, \mathcal{V}_d\, \mathcal{R}(\xi_d). \tag{20}$$

Since $\mathcal{V}_d \approx \mathcal{R}(\omega_d) + \mathcal{O}(g^2)$ (see (11)), one has

$$\mathcal{C}_d \approx \mathcal{R}(\theta_d + \omega_d + \xi_d). \tag{21}$$

The quantity $\omega_d + \xi_d$ is seen on (21) to renormalize the classical mixing angle θ_d. From (15), one deduces that it satisfies the general relation

$$\tan 2(\omega_d + \xi_d) \approx \frac{-\tan 2\omega_d \left[\frac{t_+ - t_-}{2}\ \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2}\ \frac{1}{\cos 2\omega_d}\right]}{1 + \tan^2 2\omega_d - \left[\frac{t_+ - t_-}{2}\ \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2}\ \frac{1}{\cos 2\omega_d}\right]}. \tag{22}$$

Let us suppose now that d and s are quasi-degenerate and that u and c are, at the opposite far from degeneracy. Then (see subsection 2.2), $\omega_d(\theta_c) \approx -\theta_c$, and (21) becomes $\mathcal{C}_d \approx \mathcal{R}(\theta_d - \theta_c + \xi_d(\theta_c)) =$

[5] The subscript $_L$ refers to left-handed fermions.

$\mathcal{R}(\theta_u + \xi_d(\theta_c))$. Furthermore, at the pole (16) of (15), *i.e.* when $\xi_d(\theta_c)$ becomes maximal $\xi_d = \pm\pi/4$, it is easy to show that $\omega_d + \xi_d$ gets small [6] ; θ_d is then renormalized only by a small amount [7].

4 The renormalized Cabibbo angle

4.1 The effective gauge-invariant and hermitian Lagrangian at 1-loop

After 1-loop radiative corrections to $s^0_{mL} \leftrightarrow d^0_{mL}$ and $c^0_{mL} \leftrightarrow u^0_{mL}$ have been accounted for, the kinetic terms for the first two generations of left-handed fermions, once cast into their standard form $\overline{\Psi}\overleftrightarrow{D}\Psi \equiv \frac{1}{2}\big(\overline{\Psi}D\Psi - (\overline{D\Psi})\Psi\big)$, write, in the bare mass basis

$$\mathcal{L} \in \begin{pmatrix} \bar u^0_{mL} & \bar c^0_{mL} & \bar d^0_{mL} & \bar s^0_{mL} \end{pmatrix} \left(A\not{p} - \frac{ig}{2}(A\vec{T} + \vec{T}A).\vec{W}_\mu)\,\gamma^u + \ldots\right) \begin{pmatrix} u^0_{mL} \\ c^0_{mL} \\ d^0_{mL} \\ s^0_{mL} \end{pmatrix}, \tag{24}$$

with

$$A = \left(\begin{array}{c|c} K_u & \\ \hline & K_d \end{array}\right) = \mathbb{I} + \left(\begin{array}{c|c} H_u & \\ \hline & H_d \end{array}\right). \tag{25}$$

$SU(2)_L$ gauge invariance, by requesting the replacement of the partial derivative by the covariant one, is at the origin of the gauge couplings that appear in (24). $\mathcal{L}$ is hermitian and involves the (Cabibbo rotated) $SU(2)_L$ generators $\vec{T}$

$$T^3 = \frac{1}{2}\left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right), T^+ = \left(\begin{array}{c|c} & \mathcal{C}_0 \\ \hline & \end{array}\right), T^- = \left(\begin{array}{c|c} & \\ \hline \mathcal{C}_0^\dagger & \end{array}\right), \tag{26}$$

where $\mathcal{C}_0$ is the classical Cabibbo matrix

$$\mathcal{C}_0 = \mathcal{R}(\theta_c) = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} = \mathcal{C}^\dagger_{u0}\,\mathcal{C}_{d0} = \mathcal{R}(\theta_u)^\dagger\,\mathcal{R}(\theta_d). \tag{27}$$

[6] When the pair (d,s) is quasi-degenerate and (u,c) far from degeneracy, $\omega_d \overset{(10)}{\approx} -\theta_c$ such that (16) is approximately a second degree equation in $\cos 2\theta_c$. Furthermore, one has, then (in the unitary gauge), $|h_d - h_s| \overset{m_d^2, m_s^2, p^2 \ll m_W^2}{\approx} \frac{g^2}{8\pi^2}\frac{m_s^2 - m_d^2}{m_W^2} \ll |h_u - h_c| \overset{m_u^2, m_c^2, p^2 \ll m_W^2}{\approx} \frac{g^2}{8\pi^2}\frac{m_c^2 - m_u^2}{m_W^2}$, which, added to $|h_d - h_s| \ll \frac{m_s^2 - m_d^2}{m_s^2 + m_d^2}$, enables to write the approximate solution of (16) as

$$\cos 2\theta_c \approx \frac{1}{2}(h_u - h_c)\,\frac{m_d^2 + m_s^2}{m_d^2 - m_s^2} \approx \frac{g^2}{16\pi^2}\;\frac{m_c^2 - m_u^2}{m_W^2}\;\frac{m_s^2 + m_d^2}{m_s^2 - m_d^2}. \tag{23}$$

Since the r.h.s of (23) is $\ll 1$, it corresponds to a classical θ_c itself close to maximal. Then, so does $\omega_d(\theta_c)$.

At the pole (16), $\left[\frac{t_+ - t_-}{2}\;\frac{m_d^2 + m_s^2}{m_d^2 - m_s^2}\;\frac{1}{\cos 2\omega_d}\right] = 1$ and the relation (22) becomes $\tan 2(\omega_d + \xi_d) = -1/\tan 2\omega_d$, which vanishes when ω_d becomes maximal. Then, $\omega_d + \xi_d \to 0$, *q.e.d.*

[7] Ones finds numerically from (22) and (10) that $(\omega_d + \xi_d)(\theta_c)$ only vanishes at the pole (16), *i.e.* when $\theta_c \approx -\omega_d$ is close to maximal.

4.2 The renormalized Cabibbo angle

From (24), one deduces that, in the bare mass basis, the renormalized Cabibbo matrix is, at $\mathcal{O}(g^2)$

$$\mathcal{C}^{bm}(p^2,\ldots) = \frac{1}{2}\big[\underbrace{(\mathbb{I}+H_u)}_{K_u}\mathcal{C}_0 + \mathcal{C}_0\underbrace{(\mathbb{I}+H_d)}_{K_d}\big] \tag{28}$$

which, in particular, is not unitary . Using (17), it becomes in the basis of 1-loop mass eigenstates

$$\mathfrak{C}(p^2,\ldots) = [\mathcal{V}_u\,\mathcal{R}(\xi_u)]^\dagger\,\mathcal{C}^{bm}(p^2,\ldots)\,[\mathcal{V}_d\,\mathcal{R}(\xi_d)], \tag{29}$$

Since H_u and H_d in (28) are $\mathcal{O}(g^2)$, the terms proportional to them in (29) can be calculated with the expressions of $\mathcal{R}(\xi_d)$ and $\mathcal{V}_d$ at $\mathcal{O}(g^0)$, that is, for $t_+ = 0 = t_-$; one can accordingly take in there $\mathcal{R}(\xi_d) \stackrel{(15)}{\to} \mathcal{R}(-\omega_d)$ and $\mathcal{V}_d \stackrel{(11)}{\to} \mathcal{R}(\omega_d)$, such that $\mathcal{V}_d\mathcal{R}(\xi_d) \to \mathbb{I}$. The same approximation can be done in the (u,c) sector. The resulting expression for $\mathfrak{C}$ is

$$\begin{aligned}\mathfrak{C}(p^2,\ldots) &\stackrel{\mathcal{O}(g^2)}{\approx} \mathcal{R}(\xi_u)^\dagger\,\mathcal{V}_u^\dagger\;\mathcal{C}_0\;\mathcal{V}_d\,\mathcal{R}(\xi_d) + \frac{1}{2}\big(H_u\,\mathcal{C}_0 + \mathcal{C}_0\,H_d\big)\\ &= \mathcal{C}_u^\dagger\,\mathcal{C}_d + \mathcal{O}(g^2),\end{aligned} \tag{30}$$

in the second line of which we have used (27), (20) and its equivalent for $\mathcal{C}_u$.
Let us now get an approximate expression for $\mathcal{C}_u^\dagger\,\mathcal{C}_d$ when d and s are close to degeneracy, while u and c are far from it. Then (see subsection 2.2), $\omega_d(\theta_c) \approx -\theta_c$ such that, by (11), one has $\mathcal{V}_d \approx \mathcal{R}(-\theta_c)$, which cancels the $\mathcal{C}_0 \equiv \mathcal{R}(\theta_c)$ in (30). Likewise, from the equivalent $\tan 2\omega_u = \frac{-(h_d-h_s)\sin 2\theta_c}{(h_d-h_s)\cos 2\theta_c + h_u - h_c}$ of (10) in the (u,c) sector, we deduce that, since $|h_u - h_c| \gg |h_d - h_s|$, $\omega_u \to 0$ such that, from the equivalent of (11), $\mathcal{V}_u \approx \mathbb{I}$. Also, since $\tan 2\xi_u$ is proportional to $\sin 2\omega_u$ in the equivalent of (15), ξ_u becomes small, such that $\mathcal{R}(\xi_u) \to \mathbb{I}$, too [8]. Finally, (30) becomes

$$\mathfrak{C}(p^2,\ldots) \approx \mathcal{R}(\xi_d(\theta_c)) + \mathcal{O}(g^2)\ \text{when}\ (d,s) \approx \text{ degenerate and } (u,c) \text{ far from degeneracy}. \tag{31}$$

This is our main result: the renormalized value of the Cabibbo angle finally becomes $\xi_d(\theta_c)$ as given by (15); it can become large and eventually maximal at the resonance (16). If so, since θ_c is then close to maximal, too (see footnote 6), the Cabibbo angle gets renormalized by a small amount (like θ_d (see subsection 3.2) and θ_u (see footnote 8)).

5 Summary and prospects

We have shown that, in a $SU(2)_L$ gauge model of weak interactions, 1-loop transitions between two fermions can strongly modify their mass eigenstates and generate a large mixing angle when:
* this pair is close to degeneracy;
* the other pair in the same two generations is, at the opposite, far from degeneracy.
While the classical mixing angle θ_u of the largely split pair undergoes a small renormalization, the one θ_d of the quasi-degenerate pair gets renormalized by $\xi_d(\theta_c) - \theta_c$, which play the following roles: the rotation $\mathcal{R}(\theta_c)$ casts the kinetic terms of the quasi-degenerate pair back to the unit matrix and $\mathcal{R}(\xi_d(\theta_c))$ puts its mass matrix back to diagonal. The Cabibbo angle gets accordingly renormalized from $\theta_c \equiv \theta_d - \theta_u$ to, approximately, $(\theta_d + \xi_d(\theta_c) - \theta_c) - \theta_u$, that is, up to corrections $\mathcal{O}(g^2)$, $\xi_d(\theta_c)$ itself. In the vicinity of the pole of $\tan 2\xi_d$, both θ_c and ξ_d become close to maximal. 1-loop renormalizations of θ_d, θ_c and θ_u are then small. A maximal value for the Cabibbo angle appears in these conditions as the only perturbatively stable solution (see footnote 7).

[8] $\mathcal{V}_u\mathcal{R}(\xi_u) \approx 1$, such that the renormalization of the mixing angle of the largely split pair is small.

This non-trivial effect of 1-loop radiative corrections could explain the large mixing angles observed in charged leptonic currents if the classical PMNS angles are close to fulfilling the leptonic equivalent of conditions (16) and (23). To our knowledge, it is the first time that such relations connecting masses and angles could be established on simple perturbative grounds without invoking physics beyond the standard model.

A more quantitative analysis is currently under investigation.

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